

# DAT 520 Problem Set 3 Cumulative and Conditional Probability

**Overview:** The last problem set in Module Two exposed you to dependent trials that caused the probability to shift slightly with each choice of sock from the drawer. Kind of sneaky, eh? This problem set gets you a little further into conditional probability with a direct look at Bayes’ theorem. Since this is a course in decision analysis and not a course in probability, we are going to introduce these concepts, use them once or twice by hand, but then let R and Excel do the heavy lifting for us the rest of the time. This problem set, however, requires hand calculations. All you should need is a calculator and your wits.

**Example Problem:** *Example 1*: You are flipping a fair coin four times in total, but you are flipping it in groups of two. In order to flip the coin for flips three and four, you have to get heads two times in a row for flips one and two.

## Problem Notes:

Define success: Success for this problem is flipping a head, which is *p*=50%.

Recall the binomial equation:

P open parenthesis k success in n trials parenthesis closed equals open parenthesis n divided by k parenthesis closed p to the power k open parenthesis one minus p parenthesis closed to the power n minus k equals n exclamation mark divided by open parenthesis n minus k parenthesis closed exclamation mark k exclamation mark multiplied by p to the power k multiplied by open parenthesis one minus p parenthesis closed to the power n minus k.

Recall the Law of Total Probability equation:

probability(item\_1) \* (item\_1\_%contribution) + probability(item\_2) \* (item\_2\_%contribution) + …

Recall Bayes’ theorem:

p parenthesis open A bar B parenthesis closed is equal to p parenthesis open B bar A parenthesis closed p parenthesis open A parenthesis closed divided by p parenthesis open B parenthesis closed

## Questions for Example Problem:

Q1: What is the probability of flipping heads four times in a row, **generally**?

A1: Flipping 4 heads in a row is “4 choose 4” with *p*=.5

* + Using the binomial equation - (4!/(4-4)!\*4!)\*.5^4\*(1-.5)^0
  + We calculate this as **.0625** chance of flipping four heads on four flips, *generally*.

Q2: The way this problem is set up, what is the **total probability** of getting four heads?

A2: There are two identical events of two coin flips each. So, each one is “2 choose 2” with *p*=.5 ...use the **binomial equation to calculate cumulative probability of each**:

(2!/(2-2)!\*2!) \* .5^2 \* (1-.5)^0 = .25 or 25%

We calculate this as 25% chance of flipping two heads on flips one and two. Same thing for flips three and four.

Let’s now call the first event *p*(F1&2) and the second one *p*(F3&4).

Next, **build the table**. Notice that all the % contributions have to add up to 1.

Table of contributions to total probability:

|  | P | % contrib. |
| --- | --- | --- |
| *p*(F1&2) | .25 | .50 |
| *p*(F3&4) | .25 | .50 |

Using the **law of total probability**:

* + - Total probability for two heads then two heads is (.50 x .25) + (.50 x .25) = **25%.**

Q3: What is the probability that you will get two heads on flips three and four, *conditional* on getting two heads on flips one and two?

A3: Probability of two heads on *p*(F3&4) conditional on two heads for p(F1&2).

Using Bayes Theorem, we determine that it is not very useful in this form. Here, we are looking for the probability of the second item, *p*(F3&4), given the first, *p*(F1&2). The vertical bar means “given.”

p parenthesis open F three and four bar F one and two is equal to p parenthesis open F one and two bar F three and four parenthesis closed p parenthesis open F three and four divided by p parenthesis open F one and two parenthesis closed

Which still is not very helpful. We did not calculate it this way. We did not calculate *p*(F1&2|F3&4). If we had, we would know the answer is just 1 minus the result. But we *did* calculate each part’s relation to the whole. **So** **use the form of Bayes’ theorem that uses the law of total probability as the denominator**:

P parenthesis open F three and four bar F one and two is equal to p parenthesis open F three and four parenthesis closed multiplied by percentage contribution parenthesis open F three and four divided by p parenthesis open Total parenthesis closed.

Which is *p*(F3&4)’s *proportion* to the total probability. From the table we made, we already know all of those pieces on the right-hand side:

is equal to point two five multiplied by point five zero divided by point two five is equal to fifty percent.

**Example Problem** **Interpretation:** The **conditional probability** that you will flip two heads in a row after previously flipping two heads in a row is 50%. Bear in mind that the **raw probability** of flipping two heads in a row in **independent** trials is 25% and flipping four heads in a row in independent trials is 6.25%.

## Recap of the Method Used in Example Problem:

1. Define success of an individual trial.
2. Use the binomial equation to calculate the cumulative probability of each part of the system.
3. Make a table with each part’s *p* and its percentage contribution to the whole. The percentages must add to 1.
4. Calculate the *p*(Total) using the law of total probability.
5. Use the total probability form of Bayes’ theorem to calculate the *conditional probability* for the item of interest.
6. Interpret.

**Homework Problems:**Using the example problem as reference, complete the following five problems. Turn in the work and results for each of the five problems. (Note: 4 includes a,b,c and 5 includes a,b,c,d). Note that each problem provides a hint which should guide you through the problem along with the Example Problem as a reference. Post any questions you have to the General Discussion area.

**Problem 1:** You are taking a 10 question multiple choice test. If each question has four choices and you guess on each question, what is the probability of getting one question correct? *[Hint: This is a binomial in the form of 10 choose 1 with p=.25.]*

**Problem 2:** What is the probability of getting seven questions correct? *[Hint: This is a binomial as Q1 with modified choose value.]*

**Problem 3:** What are your chances of answering seven questions correctly if you can reliably eliminate one possible answer from each question? *[Hint: This is a binomial as Q2 with a modified p value.]*

**Problem 4:** Let’s say, instead, that the test is an adaptive test; you get to answer more questions based on your previous success.

This test is structured like this:

First you have to answer three questions and if you are correct on two of them, you get to answer three more questions.

If two of **those** are correct, then you get three final questions, of which you need to get at least two correct to pass the whole test.

The test details are:

The first test, T1, has three multiple choice questions with four possible answers each (*p*=0.25 per question).

The second test, T2, has three multiple choice questions with three possible answers each (*p*=0.33 per question).

The final test, T3, has three questions that are true/false (*p*=0.50 each question).

The test questions are formed as follows:

The questions are in a language you have never seen: a mixture of Navaho, Swahili, Klingon, and Esperanto. So you have to guess on all of the questions and there are no contextual clues to eliminate any answers. This is the first one:

*'Arlogh Qoylu'pu'?*

Moja: Yel kholgo eeah.

Mbili: Floroj kreskas ĉirkaŭ mia domo. Pe'el!

Tatu: La sandviĉo estos manĝota'mo'tlhIngan maH!

Nne: 'Adeez'æ`q eeah.

(The professor sits at the front of class with a giant, sadistic grin while the students throw wads of paper at his head.)

Using the binomial probability rule, the law of total probability and Bayes’ theorem:

1. What is the probability of getting two right on each sub-exam? (T1, T2, and T3, separately.)
2. What are your overall chances of passing the entire exam?
3. What are your chances of passing T3 if you first pass T1 and T2?

### Problem Hint:

Structure your analysis.

Figure out the component probabilities: p(passing test 1), p(passing test 2), p(passing test 3).

Make a table of their proportional contributions of probability to the whole.

Calculate the total probability: p(Total).

Continue using Bayes’ theorem to calculate the probability of passing test 3 conditional on passing tests 1 and 2.

Render your interpretation. Use the interpretation in the example as a template, if you are unsure of what to say.

**Problem 5**: Now, let’s say that you know just enough of these obscure languages to translate the first question in T1:

What time is it? (Klingon)

1. (Swahili): From dawn to setting sun. (Navajo)
2. (Swahili): Flowers grow around my house (Esperanto) so all of you may come in. (Klingon)
3. (Swahili): The sandwich will be eaten (Esperanto) because we are Klingons! (Klingon)
4. (Swahili): It’s mid-afternoon. (Navajo) **[correct answer]**

Now the probability of passing T1 has changed because you only have to guess correctly on one of the two remaining questions in the first section, a one-in-two chance.

1. What is the new probability for T1?
2. Now what is the overall probability of passing the entire test?
3. And what is the probability of passing section T3, given that you have already passed sections T1 and T2?
4. The kicker: How do you explain the difference between 4c and 5c? Can you relate this to a larger context about conditional probability and making decisions?

### Problem Hint:

Compute the new probability for T1

Derived the total probability using the new value of T1

Use Bayes theorem with the updated values to compute new conditional probability of passing T3 given you have passed T1 and T2

Consider conditional probability and how T1, T2 and T3 are considered a systems